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THE METHOD OF SLOW VARIABLES IN THE PROBLEM OF THE CONTROL OF THE MOTION OF AN ELASTIC MANIPULATOR†

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A solution is proposed for the problem of synthesizing the control of a manipulator with deformable members [1–5].‡ The motion of the system is described using Routh's equation [6, 7]. The variables are the generalized coordinates, velocities and momenta of the mechanical system under consideration. A control law which depends only on the generalized coordinates and momenta of the system (but not on the generalized velocities) is constructed. Its special feature is that the variables on which it depends are slow. This will be the case when the stiffness of the members of the manipulator is sufficiently high and the generalized velocities contain high-frequency components. © 1998 Elsevier Science Ltd. All rights reserved.

The control law of a robot manipulator usually has the form of feedback involving the generalized coordinates and velocities of the mechanical system in question [1–5]. If the stiffness of the members is high, the generalized velocities of the manipulator turn out to be fast variables (they contain high-frequency components). The process of measuring such signals and generating an appropriate control signal involves difficulties, due to inertia and other non-ideal properties of the devices in the manipulator control system [8–10].

The idea underlying the solution is to make allowance for the fact that the stresses in the cross-sections of deformable members are internal forces of the mechanical system, and therefore the system has state variables which are not fast, even if the internal forces are quite high. Such a variable is, for example, the velocity of the centre of mass of a member of the manipulator. The momenta of the mechanical system are also slow variables. That is why Routh's equations are used in this paper to describe the manipulator dynamics [6, 7].

Another feature of this paper is the need to construct a universal manipulator control law, i.e. a uniform control law that will guarantee stability of a whole class of regimes rather than of a single given manipulator regime. That is to say, we are essentially dealing with stabilization of practically all possible (feasible) modes of motion compatible with the dynamics of the controlled object [8–10].

1. DYNAMICS OF THE CONTROLLED OBJECT

Each deformable member of the manipulator is considered to be a system (chain) of m ($m \geqslant 1$) absolutely rigid bodies linked by $m-1$ joints (see Fig. 1) [2, 3, 5]. It is assumed that the deformations are concentrated in the joints. The joints are general in form and may describe bending, stretching, torsion and other forms of deformation in the manipulator member. The number $m-1$ of partitions of the member is arbitrary, as is the choice of the position and shape of the cross-sections. Hence, such a finite-dimensional dynamic model enables one to make adequate allowance for the principal dynamical properties of an elastic manipulator. A comparison has been made between finite-dimensional dynamical models of mechanical systems of the type described, on the one hand, and distributed models, on the other.

As generalized coordinates of the manipulator we take the states $\varphi_1, \ldots, \varphi_n$ of the joints between its members and the states $\psi_{n+1}, \ldots, \psi_n$ of the formal joints introduced to describe the concentrated deformations of the manipulator members

$$q=(q_1,\ldots,q_N)=(\varphi,\psi), \quad \varphi=(\varphi_1,\ldots,\varphi_n), \quad \psi=(\psi_{n+1},\ldots,\psi_N) \tag{1.1}$$

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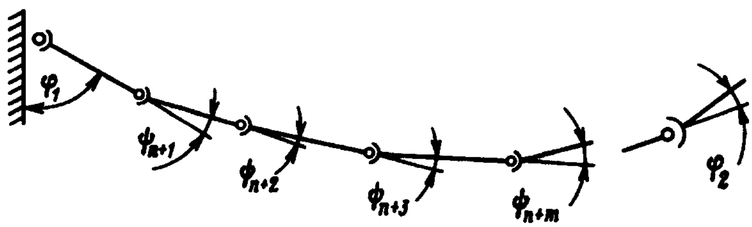

Fig. 1.

where $N = nm$ and there are no deformations when $\psi_{n+1} = \psi_{n+2} = \ldots = \psi_N = 0$ (see Fig. 1). In what follows, it will be assumed that the subscripts α and β take the values $1, 2, \ldots, n$; i, j, k take the values $n + 1, n + 2, \ldots, N$ and r, s take the values $1, 2, \ldots, N$.

Let T denote the kinetic energy of the manipulator

$$T = \tfrac{1}{2}\sum_{r,s} a_{rs}(q)\dot{q}_r\dot{q}_s \tag{1.2}$$

and let $Q_r + M_r$ be the generalized forces corresponding to (1.1). The components $Q_r = Q_r(q, \dot{q}, t)$ may describe, in particular, the weights of the manipulator members. To simplify matters, let us assume that $Q_r(q, \dot{q}, t)$, $a_{rs}(q)$ and that $a_{rs}^{-1}(q)$ are bounded together with their derivatives

$$|Q_s(q,\dot{q},t)| \leqslant C, \quad |a_{rs}(q)| \leqslant C, \ldots, \quad C = \text{const}, \quad 0 < C < \infty \tag{1.3}$$

This assumption is non-restrictive in the dynamics of mechanical systems [1–10].

Let M_α denote the controlling forces and let M_i denote forces determined by the stresses in the deformable manipulator members; it is assumed that

$$|M_\alpha| \leqslant H_\alpha, \quad H_\alpha = \text{const} > 0 \tag{1.4}$$

$$|M_i| \leqslant H_i, \quad H_i = \text{const} > 0 \tag{1.5}$$

The quantities M_α have the sense of control forces (controls) produced in each inter-member joint by the manipulator drives. To describe the physical restrictions on the possible values of these controls, we have introduced inequalities (1.4). The stresses M_i correspond to the deformations ψ_i of the manipulator members. Inequalities (1.5) have been introduced to describe the physical restrictions on the magnitudes of the admissible stresses, for eample, to keep the deformations of the manipulator members within the elastic range.

The generalized forces M_i describing the stresses in the system are given in the following form [11–13]

$$M_i = M_i^1(\psi, \dot{\psi}) = -k_i\dot{\psi}_i - k_i\lambda_i\psi_i, \quad k_i, \lambda_i = \text{const} > 0 \tag{1.6}$$

The terms $-k_i\lambda_i\psi_i$ in (1.6) may have the physical meaning of the bending moment of elastic forces while the terms $-k_i\dot{\psi}_i$ enable us to allow for the viscosity property of the deformation process and describe the corresponding forces of internal friction [11–13].

The dynamics of the mechanical system we have introduced may be described by Lagrange equations of the second kind [1–10]

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_r} - \frac{\partial T}{\partial q_r} = Q_r(q, \dot{q}, t) + M_r \tag{1.7}$$

For high stiffness of the manipulator members ($k_i \gg 1$ in (1.6)), the generalized forces M_i in system (1.7) and the accelerations $\ddot{q}_r$ may be large, and the generalized velocities $\dot{q}_r$ will be fast variables. This will follow from system (1.7), written in fully developed form [1–10].

In what follows, therefore, we will change from the generalized velocities $\dot{q}_\alpha = \dot{\varphi}_\alpha$ to new variables: generalized momenta of the mechanical system (1.7) [9, 10]

$$p_\alpha = \frac{\partial T}{\partial \dot{q}_\alpha} = \sum_\beta a_{\alpha\beta}\dot{\varphi}_\beta + \sum_i a_{\alpha i}\dot{\psi}_i \tag{1.8}$$

and we will use the equivalent Routh equations [6, 7]

$$\dot{p}_\alpha = -\frac{\partial R}{\partial \varphi_\alpha} + Q_\alpha + M_\alpha, \quad \dot{\varphi}_\alpha = \frac{\partial R}{\partial p_\alpha} \tag{1.9}$$

$$\frac{d}{dt}\frac{\partial R}{\partial \dot{\psi}_i} - \frac{\partial R}{\partial \psi_i} = -Q_i - M_i, \quad R = \sum_\alpha p_\alpha \dot{q}_\alpha - T$$

(R denotes the Routh function). The stresses M_i do not occur in the first group of Eqs (1.9). This means that p_α are slow variables (i.e. are not fast variables).

2. FORMULATION OF THE PROBLEM

The objective of the control of system (1.9) is that each coordinate q_r should vary in accordance with a certain given programme $q_r^*(t)$, that is [1–5, 8–10]

$$q_r = q_r^*(t) \tag{2.1}$$

The objective of the control of the manipulator must correspond to its dynamics, that is, the functions (2.1) must satisfy the equations of motion (1.9) and conditions (1.4) and (1.5), i.e. [8–10]

$$\dot{p}_\alpha^* \equiv -\frac{\partial R^*}{\partial \varphi_\alpha} + Q_\alpha^* + M_\alpha^*, \quad \dot{\varphi}_\alpha^* \equiv \frac{\partial R^*}{\partial p_\alpha}$$

$$\frac{d}{dt}\frac{\partial R^*}{\partial \dot{\psi}_i} - \frac{\partial R^*}{\partial \psi_i} \equiv -Q_i^* - M_i^*; \quad |M_r^*(t)| \leqslant H_r \tag{2.2}$$

The functions $M_r^* = M_r^*(t)$ in Eqs (2.2) are regarded as continuous controls corresponding to $q = q^*(t)$, $M_i^*(t) = -k_i(\psi_i^*(t) + \lambda_i\dot{\psi}_i^*(t)$. Denote the set of all such functions $q = q^*(t)$ by Φ. We also define

$$\Phi_\eta^c = (q^*(t) \in \Phi\colon |M_r^*(t)| \leqslant H_r - \eta, \ |q_r^*(t)| \leqslant c, \ |\dot{q}_r^*(t)| \leqslant c) \tag{2.3}$$

where η and c are constants, $0 < \eta \leqslant H_r$, $0 < c < \infty$ (henceforth, for simplicity, $c = C$).

The reason we have introduced Φ_η^c is to simplify the proofs of propositions in what follows. The constant η in (2.3) may be chosen to be small and c large. In that case the set Φ_η^c essentially contains all possible motions of the manipulator in which the velocities are essentially bounded and the forces M_r obey constraints (1.4) and (1.5). In that sense, the sets Φ and Φ_η^c are fairly similar to one another.

In what follows, then, we will be dealing with a fairly large subset Φ_η^c of Φ, which may be regarded as the set of all possible motions of the manipulator as a dynamical system (1.9), (1.4) and (1.5) [8–10].

The problem of synthesizing control laws for system (1.9) will be understood as that of constructing a feedback functional of the form

$$M_\alpha = M_\alpha^1(\varphi, p, \varphi^*, p^*, t) \tag{2.4}$$

which does not contain fast variables of system (1.9). The feedback (2.4) must be such that, under the substitution $q^* \to \bar{q}^*$, the new motion $\bar{q}^* = \bar{q}^*(t)$ of the closed-loop system (1.9), (2.4) will be stable provided that $\bar{q}^*(t) \in \Phi_\eta^c$. Such a control law (2.4) will be called a universal law [6–8]. The problem is to construct a universal control law (2.4) that will ensure the stability of practically all possible motions $q^*(t) \in \Phi_\eta^c$ of system (1.9).

3. SYNTHESIS OF A CONTROL LAW

Recalling a well-known scheme [8–10] for the control of a mechanical system (1.9), we will assume that the control law (2.4) has, say, the following form (the general form is (3.4))

$$M_\alpha = -H_\alpha \operatorname{sign}[p_\alpha - p_\alpha^*(t) + \lambda_\alpha(\varphi_\alpha - \varphi_\alpha^*(t))] \tag{3.1}$$

The asterisk denotes the values of the Routh variables p_α and φ_α for a given motion $q = q^*(t)$ of system

(1.9). Using (1.6) and (3.1), we write system (1.9) in the form

$$\dot{p}_\alpha = -\frac{\partial R}{\partial \varphi_\alpha} + Q_\alpha - H_\alpha \operatorname{sign}(p_\alpha - p_\alpha^* + \lambda_\alpha(\varphi_\alpha - \varphi_\alpha^*)), \quad \dot{\varphi}_\alpha = \frac{\partial R}{\partial p_\alpha}$$

$$\frac{d}{dt}\frac{\partial R}{\partial \dot{\psi}_i} - \frac{\partial R}{\partial \psi_i} = -Q_i + k_i(\dot{\psi}_i + \lambda_i \psi_i) \tag{3.2}$$

Theorem. Suppose that the various quantities in system (3.2) satisfy the inequalities

$$\lambda \leqslant \lambda_\alpha \leqslant \Lambda, \quad \bar{\lambda} \leqslant \lambda_i \leqslant \bar{\Lambda}, \quad k_i \geqslant K \tag{3.3}$$

where $K, \lambda, \Lambda, \bar{\lambda}, \bar{\Lambda}$ are certain positive constants. Then any motion $q^*(t)$ in Φ_η^c is an exponentially stable motion of system (3.2).

The proof will be given in Sections 4–6.

This theorem rigorously shows that, in principle, it is possible to control the motion of an elastic manipulator using its generalized momenta as variables. The main property of these variables is that they do not contain high-frequency components, that might appear for high stiffness of the manipulator. It thus becomes feasible to use ordinary drives and sensors as the actuators and measuring devices of the manipulator. The control law (3.1) is universal, since, according to the theorem, it guarantees the stability of any motion in Φ_η^c, that is, of practically all possible (realizable) motions of the control object under consideration [8–10].

The theorem will also hold in the general case, when the control law (3.1) and Eqs (1.6) describing the stresses in the system have the following general form

$$M_\alpha = -H_\alpha \operatorname{sign}(p_\alpha - p_\alpha^* + f_\alpha(\varphi_\alpha - \varphi_\alpha^*)), \quad M_i = F_i(\dot{\psi}_i - f_i(\psi_i)) \tag{3.4}$$

As to the functions f_α, F_i and f_i in (3.4), the following weak assumptions will suffice. The system

$$p_\alpha - p_\alpha^* + f_\alpha(\varphi_\alpha - \varphi_\alpha^*) = 0, \quad \dot{\psi}_i + f_i(\psi_i) = 0 \tag{3.5}$$

has an exponentially stable motion $q^* = (\varphi^*(t), \psi^*(t))$ (compare the special case (4.1) below); the functions f_α satisfy inequalities of type (3.3)

$$-\Lambda x \leqslant -f_\alpha(x) \leqslant -\lambda x \quad \text{for} \quad x > 0, \quad -\lambda x \leqslant -f_\alpha(x) \leqslant -\Lambda x \quad \text{for} \quad x < 0 \tag{3.6}$$

with analogous inequalities for F_i and f_i. Thus, instead of the restrictive assumption (1.6) concerning the linear structure of the stresses in the manipulator members, we have fairly weak general assumptions (3.6) concerning their properties.

The problem under consideration recalls the class of problems dealing with the stability of positions of equilibrium of mechanical systems [14]. Previously obtained conditions [14] have been replaced by the stronger conditions (3.3) in order to investigate more general (not necessarily conservative) systems, and also to guarantee stability of a wide range of motions of the manipulator (not necessarily corresponding to a position of equilibrium).

A remark is in order here as to the production of the signals p_α used in control law (3.1). It is not known how to measure these signals directly in the general case. However, the quantities p_α may be expressed as functions (1.8) $p_\alpha = p_\alpha(q, \dot{q})$, whose arguments q_r, $\dot{q}_r$ are signals that are usually accessible to measurement. The main special feature of the function $p_\alpha = p_\alpha(q, \dot{q})$ is that it does not contain high-frequency components, despite the fact that the arguments $\dot{q}_r$ may contain them.

This property may be made the basis of a method for the special processing (estimation or observation) of measurable signals $(q_r, \dot{q}_r)$ with a view to improving the accuracy of the output signal $p_\alpha = p_\alpha(q, \dot{q})$. In that connection, the following detail is worthy of attention. The coefficients $a_{rs} = a_{rs}(q)$ in expression (1.8) for the momenta may vary, depending on what scheme is chosen to describe the deformation of elastic members. The following factors are arbitrary here: the number m of elements into which the manipulator member is divided, the size of the elements and the positions and types of joints linking the elements. This arbitrariness may also be used in order to improve the accuracy with which the output signal $p_\alpha = p_\alpha(q, \dot{q})$ is estimated.

4. OUTLINE OF PROOF OF THE THEOREM

Following the approach described in [8–10], let us consider the motion of system (3.2) in a regime of the form

$$p_\alpha = p_\alpha^* - \lambda_\alpha(\varphi_\alpha - \varphi_\alpha^*), \quad \dot{\psi}_i = \dot{\psi}_i^* - \lambda_i(\psi_i - \psi_i^*) \tag{4.1}$$

(in the general case, Eqs (4.1) have the form (3.5)). System (4.1), expressed in terms of differences $\xi_r = q_r - q_r^*$ and taking (1.8) into consideration, has the form

$$\sum_\beta a_{\alpha\beta}\dot{\xi}_\beta = -\lambda_\alpha\xi_\alpha - \sum_r (a_{\alpha r} - a_{\alpha r}^*)\dot{q}_r^* + \sum_i a_{\alpha i}\lambda_i\xi_i, \quad \dot{\xi}_i = -\lambda_i\xi_i \tag{4.2}$$

Lemma. The motion $\xi = 0$ of system (4.2) will be exponentially stable if $q^*(t) \in \Phi_\eta^c$ and the following of inequalities (3.3) are satisfied

$$\lambda \leqslant \lambda_\alpha \tag{4.3}$$

The proof will be given in Section 5.

The main idea is thus to show that system (3.2) indeed admits of a motion (4.1). In that case, it will follow from the stability of the motion $q = q^*(t)$ of system (4.1) that the motion $q = q^*(t)$ in system (3.2) is also stable—the conclusion of the theorem.

We will show that system (3.2) can be reduced to a motion (4.1). To that end, we will write (3.2) in the following developed form [6, 7]

$$\begin{aligned}
\dot{p}_\alpha &= O_\alpha - H_\alpha \operatorname{sign}(p_\alpha - p_\alpha^* + \lambda_\alpha(\varphi_\alpha - \varphi_\alpha^*)) \\
\dot{\varphi}_\alpha &= \sum_\beta b_{\alpha\beta}(q)\left(p_\beta - \sum_i a_{\beta i}(q)\dot{\psi}_i\right) \\
\sum_i A_{ij}\ddot{\psi}_j &= O_i - \sum_\alpha \dot{p}_\alpha\gamma_{\alpha i} - k_i(\dot{\psi}_i + \lambda_i\psi_i)
\end{aligned} \tag{4.4}$$

where

$$O_\alpha = Q_\alpha - \frac{\partial R}{\partial q_\alpha}, \quad O_i = -\sum_\alpha p_\alpha\dot{\gamma}_{\alpha i} - \sum_j \dot{A}_{ij}\dot{q}_j + Q_i - \frac{\partial R}{\partial q_i} \tag{4.5}$$

$$A_{ij}(q) = a_{ij}(q) - \sum_{\alpha,\beta} b_{\alpha\beta}a_{\alpha j}a_{\beta j} \tag{4.6}$$

$$\det a \neq 0, \quad a = \| a_{\alpha\beta}(q)\|, \quad b = \| b_{\alpha\beta}(q)\| = a^{-1} \tag{4.7}$$

$$\gamma_{\alpha i} = \sum_\beta b_{\alpha\beta}(q)a_{\beta i}(q) \tag{4.8}$$

We introduce the deviations of the motion of system (4.4) from regime (4.1)

$$\chi_\alpha = \Delta p_\alpha + \lambda_\alpha\xi_\alpha, \quad \chi_i = \dot{\xi}_i + \lambda_i\xi_i, \quad \Delta p_\alpha = p_\alpha - p_\alpha^*(t), \quad \xi_r = q_r - q_r^*(t) \tag{4.9}$$

On this basis we introduce a general measure for the deviation of the motion of system (4.4) from regime (4.1), characterized by functions of the form

$$\Pi = \frac{1}{2}\sum_\alpha \chi_\alpha^2, \quad G = \frac{1}{2}\sum_{i,j} A_{ij}(q)\chi_i\chi_j \tag{4.10}$$

Π and G are the components of a Lyapunov vector function [15]; they are positive-definite functions of the variables χ_α and χ_i, respectively. For G this follows from the inequalities [6, 7]

$$r_1^2\sum_i \chi_i^2 \leqslant G \leqslant r_2^2\sum_i \chi_i^2 \tag{4.11}$$

where $0 < r_1 \leqslant r_2 < \infty$, as follows from conditions (1.3).

The main thrust of the proof involves verification of the equalities $\Pi = 0$ and $G = 0$. Hence, taking (4.10) and (4.11) into account, one immediately derives the equalities $\chi_i = 0$. This means that system (4.4) is moving in mode (4.1). The motion $q = q^*$ of system (4.1) is (by the lemma) exponentially stable. Hence it follows that the motion $q = q^*$ of system (4.4) is also exponentially stable, which proves the theorem. The equalities $\Pi = 0$ and $G = 0$ are proved by constructing the derivatives of the Lyapunov functions Π, G along trajectories of system (4.4) and showing that they decrease to zero on motions of system (4.4) (Section 6).

5. PROOF OF THE LEMMA

The lemma follows from the stability of the motion $\xi = 0$ of system (4.2) with respect to the variable $\xi_\alpha = 0$. To prove this fact we use the Lyapunov function

$$g = \frac{1}{2}\sum_{\alpha,\beta} a_{\alpha\beta}(q)\xi_\alpha\xi_\beta \tag{5.1}$$

which is a positive-definite function of ξ_α [6, 7].

The motion of system (4.4) will be considered in the domain

$$|q_r| \leqslant c, \quad |\dot{q}_r| \leqslant c \tag{5.2}$$

which is analogous to (2.3). It follows from (1.3) that the following inequalities, analogous to (4.11), hold in (5.2) [6–10]

$$\rho_1^2 \sum_\alpha \xi_\alpha^2 \leqslant g \leqslant \rho_2^2 \sum_\alpha \xi_\alpha^2, \quad \rho_i = \text{const}, \quad 0 < \rho_1 \leqslant \rho_2 < \infty \tag{5.3}$$

The expression for the derivative of g along trajectories of system (4.2) implies the limit

$$\dot{g} \leqslant -\sum_\alpha \lambda\xi_\alpha^2 + \sum_\alpha |\xi_\alpha| \left[\frac{1}{2}\sum_\beta |\dot{a}_{\alpha\beta}||\xi_\beta| + \sum_r |\Delta a_{\alpha r}||\dot{q}_r^*| + \sum_i |a_{\alpha i}|\bar{\Lambda}|\xi_i|\right] \tag{5.4}$$

$$\lambda \leqslant \lambda_\alpha \leqslant \Lambda, \quad \bar{\lambda} \leqslant \lambda_i \leqslant \bar{\Lambda} \tag{5.5}$$

It follows from (1.3) that

$$|a_{\alpha i}(q)| \leqslant C, \; \Delta a_{\alpha i} = a_{\alpha i}(q^* + \xi) - a_{\alpha i}(q^*) = \sum_s \frac{\partial a_{\alpha i}}{\partial q_s}\xi_s \leqslant \sum_s C|\xi_s| \tag{5.6}$$

Taking (2.3), (5.2) and (5.6) into consideration, we deduce from (5.4) that

$$\dot{g} \leqslant -\sum_\alpha \lambda\xi_\alpha^2 + S\left[\frac{1}{2}SCNc + (S + \bar{S})CNc + \bar{S}C\bar{\Lambda}\right] \tag{5.7}$$

$$S = \sum_\alpha |\xi_\alpha|, \quad \bar{S} = \sum_j |\xi_j|$$

It follows from (5.7) that

$$\dot{g} \leqslant -\frac{\lambda g}{\rho_2^2} + \frac{n\sqrt{g}}{\rho_1}\left[\frac{n\sqrt{g}}{\rho_1}\frac{3}{2}CNc + \bar{S}C(Nc + \bar{\Lambda})\right] \tag{5.8}$$

where we have used the inequalities $\Sigma_\alpha \xi_\alpha^2 \leqslant -g/\rho_2^2$, $|\xi_\alpha| \leqslant \sqrt{(g)}/\rho_1$ derived from (5.3).

Let δ be a number defining the initial data for system (4.4)

$$|\xi_s(0)| \leqslant \delta, \quad |\dot{\xi}_s(0)| \leqslant \delta \tag{5.9}$$

Then, using (4.2), we deduce from (5.8) that

$$\dot{g} \leqslant -\frac{\lambda g}{\rho_2^2} + \frac{n\sqrt{g}}{\rho_1}\left[\frac{n\sqrt{g}}{\rho_1}\frac{3}{2}CNc + C(Nc+\bar{\Lambda})\sum_i \delta\exp(-\lambda_i t)\right] \tag{5.10}$$

Using the condition (see condition (4.3) of the lemma)

$$-\lambda^1 = -\frac{\lambda}{\rho_2^2} + \frac{3}{2}\left(\frac{n}{\rho_1}\right)^2 CNc < 0 \tag{5.11}$$

we can write inequality (5.10) in the form

$$\sqrt{g}\left[\frac{d}{dt}(2\sqrt{g}) + \lambda^1\sqrt{g} - b\exp(-\bar{\lambda}t)\right] \leqslant 0 \tag{5.12}$$

The motion $g(t) = 0$ of system (5.12) (in the class of non-negative absolutely continuous functions $g(t)$) is exponentially stable [8–10]. Hence, in view of (5.3), we deduce that the motion $\xi = 0$ of system (4.2) is also exponentially stable with respect to the variable ξ_α, which proves the lemma.

6. PROOF OF THE THEOREM

The derivatives of the Lyapunov functions Π and G along trajectories of system (4.4) (which is equivalent to (3.2)) take the form

$$\begin{aligned}
\dot{\Pi} &= \sum_\alpha \chi_\alpha(\Delta O_\alpha + \Delta M_\alpha + \lambda_\alpha\dot{\xi}_\alpha) \\
\dot{G} &= \sum_i \chi_i\left\{\frac{1}{2}\sum_k \dot{A}_{ik}\chi_k + \left[-\sum_k \Delta A_{ik}\ddot{q}_k^* + \Delta O_i - \sum_\alpha \Delta\dot{p}_\alpha\gamma_{\alpha i}^* + \right.\right. \\
&\left.\left. -\sum_\alpha \dot{p}_\alpha\Delta\gamma_{\alpha i} + \Delta M_i\right] + \sum_k A_{ik}\lambda_k\dot{\xi}_k\right\}
\end{aligned} \tag{6.1}$$

These formulae are established taking (4.9) and the identities (2.2) of type (4.4) into consideration. System (6.1) can be rewritten as follows:

$$\begin{aligned}
\dot{\Pi} &= \sum_\alpha \chi_\alpha(\Delta O_\alpha + \lambda_\alpha\dot{\xi}_\alpha) + \sum_\alpha \chi_\alpha\Delta M_\alpha \\
\dot{G} &= \sum_i \chi_i\left\{\sum_k \dot{A}_{ik}\chi_k/2 + \left[-\sum_k \Delta A_{ik}\ddot{q}_k^* + \Delta O_i + \sum_\alpha \lambda_\alpha\dot{\xi}_\alpha\gamma_{\alpha i}^* + \right.\right. \\
&\left.\left. -\sum_\alpha (O_\alpha + M_\alpha)\Delta\gamma_{\alpha i}\right] + \sum_k A_{ik}\lambda_k\dot{\xi}_k\right\} + \sum_i \chi_i\Delta M_i + \sigma
\end{aligned} \tag{6.2}$$

where

$$\sigma = -\sum_i \chi_i\sum_\alpha (\Delta\dot{p}_\alpha + \lambda_\alpha\dot{\xi}_\alpha)\gamma_{\alpha i}^* \tag{6.3}$$

and the equations $p_\alpha = O_\alpha + M_\alpha$ of system (4.4) have been taken into consideration.

The right-hand sides of system (6.2) will now be majorized in the domain (5.2), taking (2.3) into consideration. This is done using the following estimates of type (5.6)

$$|\dot{A}_{ik}| < CNc,\quad |\ddot{q}_k^*| \leqslant C,\quad |\gamma_{\alpha i}^*| \leqslant C,\quad |O_\alpha + M_\alpha| \leqslant C + H_\alpha,\quad |A_{ik}| \leqslant C$$

$$\Delta O_\alpha \leqslant C(n\sqrt{\Pi} + (N-n)\sqrt{G} + S(1+\Lambda) + \bar{S}(1+\bar{\Lambda}))$$

which follow from the fact that the functions in (6.20 are smooth (this in turn follows from (1.3) by way of (4.5)–(4.8)). The relations $\chi_\alpha\Delta M_\alpha \leqslant -\eta|\chi_\alpha|$, $\chi_i\Delta M_i = -k_i\chi_i^2$, which follow from (1.6), (3.1), (4.9) and (2.3), have been taken into account. Relations (4.10) and inequalities (4.11) are used. This enables us, starting from system (6.2), to construct a system of inequalities

$$\dot\Pi \leqslant \sqrt{\Pi}\{a^1\sqrt{\Pi}+a^2\sqrt{G}+a^2S+a^3\bar{S}\}-\eta\sqrt{2\Pi}$$
$$\dot G \leqslant \sqrt{G}\{b^1\sqrt{\Pi}+b^2\sqrt{G}+b^3S+b^4\bar{S}\}-G\frac{k}{r^2}+\sigma \tag{6.4}$$

where a_i, $b_i \geqslant 0$ are constants, $0 < k \leqslant k_i$.

If the Lyapunov functions (5.1) are used, as well as the formula $z = \Sigma_k\xi_k^2/2$, system (6.4) becomes

$$\dot\Pi \leqslant -\sqrt{\Pi}\{\eta\sqrt{2}-A^1\sqrt{\Pi}-A^2\sqrt{G}-A^3\sqrt{g}-A^4\sqrt{z}\}$$
$$\dot G \leqslant B^1\Pi-\left(\frac{k}{r^2}-B^2\right)G+B^3g+B^4z+\sigma \tag{6.5}$$

where use has been made of the estimates $S \leqslant \sqrt{(g)}n/\rho_1$, $\bar{S} \leqslant \sqrt{(2z)}(N-n)$ and of $2\sqrt{(\Pi G)} \leqslant \Pi + G$.

The following inequalities hold for the derivatives $\dot g$ and $\dot z$ along trajectories of system (4.4)

$$\dot g \leqslant \left\{-\lambda\frac{g}{\rho_2^2}+Cn\frac{\sqrt{g}}{\rho_1}\left[\frac{2}{3}nNc\frac{\sqrt{g}}{\rho_1}+(Nc+\bar\Lambda)\bar{S}\right]\right\}+\sum_{\alpha,\beta}\frac{1}{2}a_{\alpha\beta}\xi_\beta\chi_\alpha$$
$$\dot z = -\sum_k \lambda_k\xi_k^2+\sum_k \xi_k\xi_k \leqslant -2\bar\lambda z+\sum_k \sqrt{2z}\frac{\sqrt{G}}{r_1} \tag{6.6}$$

where we have used the estimate (5.8) for $\dot g$ along trajectories of (4.4) in the motion of the system in mode (4.1), where $\chi_\alpha \equiv 0$, $\chi_k \equiv 0$. In view of (6.6), system (6.5) becomes

$$\dot\Pi \leqslant -\sqrt{\Pi}\{n\sqrt{2}-A^1\sqrt{\Pi}-A^2\sqrt{G}-A^3\sqrt{g}-A^4\sqrt{z}\}$$
$$\dot G \leqslant B^1\Pi-\left(\frac{k}{r^2}-B^2\right)G+B^3g+B^4z+\sigma$$
$$\dot g \leqslant C^1\Pi-(\lambda g/\rho_2^2-C^3)g+C^4z,\quad \dot z \leqslant D^2G-(2\bar\lambda-D^4)z \tag{6.7}$$

We will further show that the motion $\Pi = 0$, $G = 0$, $g = 0$, $z = 0$ of system (6.7) is exponentially stable. This will imply the exponential stability of the motion $q = q^*(t)$ of system (4.4), as claimed by the theorem. Below we will show that:

(i) for small initial deviations

$$\Pi(0) \leqslant \delta,\ G(0) \leqslant \delta,\ g(0) \leqslant \delta,\ z(0) \leqslant \delta \tag{6.8}$$

the variable Π of system (6.7) will reach zero in a fairly short time $t = t_1$ and then remain equal to zero;

(ii) the motion $\Pi = 0$, $G = 0$, $g = 0$, $z = 0$ of system (6.7) with $\sigma = 0$ is exponentially stable (the parameters k, λ, Λ, λ^-, Λ^- may be chosen accordingly);

(iii) system (6.7) will also be exponentially stable when $\sigma \not\equiv 0$ (when the properties of this term are taken into account), whence it will follow that the motion $q = q^*(t)$ is exponentially stable.

Accordingly, let us consider the motion (6.7) in the domain

$$\Pi \leqslant \delta_1,\ G \leqslant \delta_1,\ g \leqslant \delta_1,\ z \leqslant \delta_1 \tag{6.9}$$

where δ_1 satisfies the inequality

$$\eta\sqrt{2}-A^1\sqrt{\delta_1}-A^2\sqrt{\delta_1}-A^3\sqrt{\delta_1}-A^4\sqrt{\delta_1} \geqslant \eta\sqrt{2}/2 \tag{6.10}$$

Then system (6.7) may be written in the form

$$\dot\Pi \leqslant -\sqrt{\Pi}\eta\sqrt{2}/2,\quad \dot G \leqslant \gamma,\quad \dot g \leqslant \gamma,\quad \dot z \leqslant \gamma,\quad \gamma = \text{const} \geqslant 0 \tag{6.11}$$

The function Π in (6.11) will satisfy the following relations [8–10]

$$\sqrt{\Pi(t)} \leqslant \sqrt{\Pi(0)} - t\eta\sqrt{2}/4 \quad \text{for} \quad t \leqslant t_1, \quad \sqrt{\Pi(t)} \equiv 0 \quad \text{for} \quad t > t_1 \tag{6.12}$$
$$t_1 = 4\sqrt{\Pi(0)}/(\eta\sqrt{2})$$

In view of (6.12), we can rewrite system (6.11) in the form

$$\sqrt{\Pi(t)} \leqslant \sqrt{\Pi(0)}, \quad G(t) \leqslant G(0) + \gamma t, \quad g(t) \leqslant g(0) + \gamma t, \ldots \tag{6.13}$$

Choose δ in (6.8) to satisfy the condition $G(2t_1) \leqslant \delta_1, \ldots$, that is, the condition $\delta + 2\gamma(4\sqrt{\delta}/(\eta\sqrt{2})) \leqslant \delta_1$. Then the values of the variables G, g, and z will not leave domain (6.9) for $t \leqslant 2t_1$. This means that when $t \leqslant 2t_1$ system (6.11) will move along trajectories of the original system (6.7).

We will show that the motion $\Pi = 0$, $G = 0$, $g = 0$, $z = 0$ of system (6.7) will continue to remain in domain (6.9); even more, it will be exponentially stable. To that end, we write system (6.7) for $t = t_1 + 0$, using (6.12), as follows:

$$\begin{aligned} &\Pi \equiv 0, \quad \dot{G} \leqslant -(k/r_2^2 - B^2)G + B^3 g + B^4 z + \sigma \\ &\dot{g} \leqslant -(\lambda g/\rho_2^2 - C^3)g + C^4 z, \quad \dot{z} \leqslant D^2 G - (2\bar{\lambda} - D^4)z \end{aligned} \tag{6.14}$$

We now consider the sum

$$Y = \mu^2 G + \mu g + z, \quad \mu = \text{const} > 0 \tag{6.15}$$

and its derivative along trajectories of system (6.14)

$$\begin{aligned} &\dot{Y} \leqslant \mu^2(-(k/r_2^2 - B^2)G + B^3 g + B^4 z + \sigma) + \\ &+\mu(-(\lambda g/\rho_2^2 - C^3)g + C^4 z) + D^2 G - (2\bar{\lambda} - D^4)z \end{aligned} \tag{6.16}$$

With the parameters k, λ, Λ, $\bar{\lambda}$, $\bar{\Lambda}$ suitably chosen, the coefficients of system (6.16) will satisfy the inequalities

$$\mu^2(k/r_2^2 - B^2) > D^2, \quad \mu(\lambda/\rho_2^2 - C^3) > \mu^2 B^3, \quad 2\bar{\lambda} - D^4 > \mu C^4 + \mu^2 B^4 \tag{6.17}$$

This may be verified, say, for $\Lambda = 2\lambda$, $\bar{\Lambda} = 2\bar{\lambda}$, $\bar{\lambda} > 4D^4$, $\lambda/\rho_2^2 - C^3(2\lambda) = C^3(2\bar{\lambda})$ and sufficiently large μ and $1/K$. In view of (6.17), inequality (6.16) may be written as

$$\dot{Y} \leqslant -EY + \mu^2\sigma, \quad E = \text{const} > 0 \tag{6.18}$$

that is, we have verified the stability of system (6.18) (which implies that of system (6.8)) for $\sigma = 0$.

We will now show that system (6.18) will also be stable for $\sigma \neq 0$. To that end, we consider the general solution of inequality (6.18)

$$Y(t) \leqslant [Y(t_1) + \exp(E(\tau - t_1))I(\tau)|_{t_1}^{t} + \int_{t_1}^{t} E\exp(E(\tau - t_1))I(\tau)d\tau]\exp(-E(t - t_1)) \tag{6.19}$$

where, due to (6.3)

$$I(t) = \int_{t_1}^{t} \mu^2\sigma d\tau = \int_{t_1}^{t} \mu^2\left[-\sum_{i}\sum_{\alpha} \chi_i\gamma_{\alpha_i}^{*}(\Delta\dot{p}_\alpha + \lambda_\alpha\dot{\xi}_\alpha)\right]d\tau \tag{6.20}$$

For $t \geqslant t_1$ we have the identity

$$I(t) \equiv 0, \quad t \geqslant t_1 \tag{6.21}$$

which may be verified by integration by parts, with $d\upsilon = \Sigma_\alpha(\Delta p_\alpha + \lambda_\alpha\xi_\alpha)d\tau$. One also takes into account that the derivatives of the functions χ_i and $\gamma_{\alpha i}^{*}$ are bounded, due to conditions (1.3), in the domain (2.3), (5.2), and that $\upsilon \equiv 0$ for $t \geqslant t_1$, which follows from (6.12).

In view of (6.21), we can rewrite inequality (6.19) as follows:

$$Y(t) \leqslant Y(t_1)\exp(-E(t-t_1)), \quad t \geqslant t_1 \tag{6.22}$$

Hence it follows that the motion $\Pi = 0, G = 0, g = 0, z = 0$ of system (6.7) is exponentially stable, that is, the motion $q = q^*(t)$ of the original system (4.4) is exponentially stable. This completes the proof of the theorem.

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